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LARGE DEFLECTIONS OF HETEROGENEOUS ANISOTROPIC RECTANGULAR PLATES

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Abstract—An approximate solution is presented for large deflections of clamped, uniformly loaded, unsymmetrically laminated, anisotropic, rectangular plates. Expressing the load and displacements in the form of series, the von Karman-type nonlinear differential equations and immovable boundary conditions are reduced to a series of linear partial differential equations and boundary conditions. The solution obtained by successive approximations can reduce to some existing solutions for large deflections of homogeneous plates. Numerical results based on the first three terms of the truncated series are graphically presented for unsymmetrical cross-ply and angle-ply plates having various values of fiber-reinforced material, number of layers, and aspect ratio. The results in small deflections of coupled laminates are compared with available data.

NOTATION

A, B, D	stiffness matrices of plate
A_{ij}, B_{ij}, D_{ij}	stiffness coefficients of plate
a, b	plate length and width along x and y directions
a_i, b_i, c_i	nondimensional stiffness coefficients of plate
AP, CP	angle-ply and cross-ply plates
BO, GL, GR	boron-epoxy, glass-epoxy and graphite-epoxy
$C_{ij}^{(k)}$	stiffness coefficients of a layer
E_L, E_T	tensile moduli in filament and transverse directions
F, G, H	polynomials
G_{LT}	shear modulus
h	plate thickness
L_i	differential operators
M, N	stress couple and stress resultant matrices
M_x, M_y, M_{xy}	stress couples
M_ζ, M_η, $M_{\zeta\eta}$	nondimensional stress couples
N_x, N_y, N_{xy}	stress resultants
N_ζ, N_η, $N_{\zeta\eta}$	nondimensional stress resultants
n	number of layers in a plate
Q	nondimensional load parameter
q	uniform transverse load
q_i	coefficients in load expansion
R_{nij}, S_{nij}, T_{nij}	coefficients in polynomials
U, V, W	nondimensional displacement components in ζ, η, z directions
u^0, v^0, w	displacement components in x, y, z directions
u_i, v_i, w_i	variable coefficients in series expansions for U, V and W
W_0	nondimensional central deflection
w_0	central deflection

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x, y, z	Cartesian coordinates
Z_{nij}	S_{nij} or T_{nij}
δ_i	nondimensional midsurface strains
ε	midsurface strain matrix
$\varepsilon_x^0, \varepsilon_y^0, \varepsilon_{xy}^0$	midsurface strains
ζ, η	nondimensional Cartesian coordinates
λ	aspect ratio
κ	bending curvature matrix
$\kappa_x, \kappa_y, \kappa_{xy}$	bending curvatures
ν_{LT}	Poisson's ratio
$(\ )_{,i}$	partial differentiation with respect to $i = \zeta, \eta$.

INTRODUCTION

The elastic problem of unsymmetrically laminated rectangular plates has received considerable attention. Reissner and Stavsky[1] have shown that a coupling occurs between bending and stretching for two-layer, angle-ply plates. Based on Kirchhoff's hypothesis, Stavsky[2] has developed a linear theory for coupled laminates and presented solutions for cylindrical bending and uniform distribution of stress resultants and couples. Whitney and Leissa[3] have derived the von Karman-type large-deflection equations in terms of displacements for composite plates and obtained linear solutions for sinusoidal transverse load, flexural vibration and stability of unsymmetrical cross-ply and angle-ply plates with simply supported edges. Using the same set of equations and boundary conditions as above, a double Fourier series solution has been given by Whitney[4] for the bending of plates under transverse load. Expressing the governing differential equations in terms of transverse deflection and stress function, Whitney and Leissa[5] have presented double Fourier series solutions for simply supported plates under uniform pressure, transverse vibration, and buckling. Using the linearized equations[3] and multiple Fourier method, Whitney[6] has also discussed the bending, buckling and vibration of composite plates with various sets of boundary conditions. The bending of unsymmetrically laminated orthotropic plates with simply supported edges has been treated by Holston[7] expanding the transverse load and stress function into double Fourier series. Applying the method analogous to Levy's solution for isotropic plates, Kan and Ito[8, 9] have recently investigated the bending problem of coupled laminates with simple support on two edges and general boundary conditions on the other two edges. The other related work based on the linear theory can be found elsewhere. There have been a few investigations of composite plates based on von Karman-type large-deflection theory. Assuming a particular form for the transverse deflection, Pao[10] has presented a solution for simple bending of unsymmetrically laminated anisotropic plates. Applying the Galerkin method, Bennett[11] has studied the nonlinear vibration of unsymmetrical angle-ply plates. A double Fourier series solution in terms of appropriate beam eigenfunctions has been obtained by the writer and Prabhakara[12] for the postbuckling of unsymmetrically laminated anisotropic plates.

This paper is analytically concerned with the large-deflection behavior of a heterogeneous anisotropic rectangular plate. The plate is clamped along its edges and subjected to uniform transverse load. The type of plate under consideration consists of n layers of orthotropic sheets perfectly bonded together. Each layer has arbitrary thickness, elastic properties and orientation of orthotropic axes with respect to the plate axes. A solution for the boundary-problem is formulated on the basis of the perturbation technique and specified for unsymmetric angle-ply and cross-ply plates. Unsymmetric angle-ply and cross-ply plates are assumed to consist of an even number of layers all of the same thickness and the same elastic properties. The orthotropic axes of symmetry in each ply are alternately oriented at angles of $+\theta$ and $-\theta$ to the plate axes in the former and at 0° and 90° in the latter.

GOVERNING DIFFERENTIAL EQUATIONS

Consider a rectangular plate of length $2a$ in the x direction, width $2b$ in the y direction and thickness h in the z direction. The origin of the coordinate system is chosen to coincide with the center of the midplane of the undeformed plate. The type of plate under discussion consists of n layers of orthotropic sheets perfectly bonded together. Each layer has arbitrary thickness, elastic properties, and orientation of orthotropic axes with respect to the plate axes.

The differential equations governing the large-deflection behavior of the plate under uniformly distributed lateral load q per unit area can be derived from the classical non-linear theory of elastic plates. Let membrane stress resultants, N_x, N_y, N_{xy}, stress couples, M_x, M_y, M_{xy}, midsurface strains, ε_x^0, ε_y^0, ε_{xy}^0, and bending curvatures, κ_x, κ_y, κ_{xy}, be defined as usual in the classical theory of thin homogeneous plates. The constitutive relations for the composite plate can be written in the matrix notation

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} \varepsilon^0 \\ \kappa \end{bmatrix} \tag{1}$$

where

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} C_{ij}^{(k)}(1, z, z^2)\,\mathrm{d}z \qquad (i, j = 1, 2, 6) \tag{2}$$

in which $C_{ij}^{(k)}$ are the anisotropic stiffness coefficients of the kth layer. If the two in-plane displacements and transverse deflection at the midsurface in the x, y, z directions are denoted by u^0, v^0 and w, respectively, the governing differential equations[3] in the present case can be written in the nondimensional form.

$$\begin{aligned}
&\lambda L_1 U + \lambda L_2 V - L_3 W = -W_{,\zeta} L_1 W - \lambda W_{,\eta} L_2 W \\
&\lambda L_2 U + \lambda L_4 V - L_5 W = -W_{,\zeta} L_2 W - \lambda W_{,\eta} L_4 W \\
&\lambda L_3 U + \lambda L_5 V - L_6 W = -\lambda^4 Q - (\lambda U_{,\zeta} + W_{,\zeta}{}^2/2) L_7 W \\
&\qquad - \lambda^2 (V_{,\eta} + W_{,\eta}{}^2/2) L_8 W - \lambda(\lambda U_{,\eta} + V_{,\zeta} + W_{,\zeta} W_{,\eta}) L_9 W \\
&\qquad - W_{,\zeta} L_3 W - \lambda W_{,\eta} L_5 W - 2\lambda(b_3 - b_4)(W_{,\zeta\eta}{}^2 - W_{,\zeta\zeta} W_{,\eta\eta}).
\end{aligned} \tag{3}$$

In equations (3), the comma denotes partial differentiation with respect to the corresponding coordinate and the nondimensional parameters and constants are defined by

$$\begin{aligned}
&U = bu^0/h^2, && V = bv^0/h^2, && W = w/h \\
&Q = qb^4/h^3 A_{11}, && \zeta = x/a, && \eta = y/b, \quad \lambda = a/b \\
&a_1 = A_{16}/A_{11}, && a_2 = A_{66}/A_{11}, && a_3 = (A_{12} + A_{66})/A_{11} \\
&a_4 = A_{26}/A_{11}, && a_5 = A_{22}/A_{11}, && a_6 = A_{12}/A_{11} \\
&b_1 = B_{11}/hA_{11}, && b_2 = B_{16}/hA_{11}, && b_3 = B_{12}/hA_{11} \\
&b_4 = B_{66}/hA_{11}, && b_5 = B_{26}/hA_{11}, && b_6 = B_{22}/hA_{11} \\
&c_1 = D_{11}/h^2 A_{11}, && c_2 = D_{16}/h^2 A_{11}, && c_3 = (D_{12} + 2D_{66})/h^2 A_{11} \\
&c_4 = D_{26}/h^2 A_{11}, && c_5 = D_{22}/h^2 A_{11}
\end{aligned} \tag{4}$$

and the nondimensional linear differential operators by

$$
\begin{aligned}
L_1(\ ) &= (\ )_{,\zeta\zeta} + 2\lambda a_1(\ )_{,\zeta\eta} + \lambda^2 a_2(\ )_{,\eta\eta} \\
L_2(\ ) &= a_1(\ )_{,\zeta\zeta} + \lambda a_3(\ )_{,\zeta\eta} + \lambda^2 a_4(\ )_{,\eta\eta} \\
L_3(\ ) &= b_1(\ )_{,\zeta\zeta\zeta} + 3\lambda b_2(\ )_{,\zeta\zeta\eta} + \lambda^2(b_3 + 2b_4)(\ )_{,\zeta\eta\eta} + \lambda^3 b_5(\ )_{,\eta\eta\eta} \\
L_4(\ ) &= a_2(\ )_{,\zeta\zeta} + 2\lambda a_4(\ )_{,\zeta\eta} + \lambda^2 a_5(\ )_{,\eta\eta} \\
L_5(\ ) &= b_2(\ )_{,\zeta\zeta\zeta} + \lambda(b_3 + 2b_4)(\ )_{,\zeta\zeta\eta} + 3\lambda^2 b_5(\ )_{,\zeta\eta\eta} + \lambda^3 b_6(\ )_{,\eta\eta\eta} \\
L_6(\ ) &= c_1(\ )_{,\zeta\zeta\zeta\zeta} + 4\lambda c_2(\ )_{,\zeta\zeta\zeta\eta} + 2\lambda^2 c_3(\ )_{,\zeta\zeta\eta\eta} + 4\lambda^3 c_4(\ )_{,\zeta\eta\eta\eta} + \lambda^4 c_5(\ )_{,\eta\eta\eta\eta} \\
L_7(\ ) &= (\ )_{,\zeta\zeta} + 2\lambda a_1(\ )_{,\zeta\eta} + \lambda^2 a_6(\ )_{,\eta\eta} \\
L_8(\ ) &= a_6(\ )_{,\zeta\zeta} + 2\lambda a_4(\ )_{,\zeta\eta} + \lambda^2 a_5(\ )_{,\eta\eta} \\
L_9(\ ) &= a_1(\ )_{,\zeta\zeta} + 2\lambda a_2(\ )_{,\zeta\eta} + \lambda^2 a_4(\ )_{,\eta\eta}.
\end{aligned} \tag{5}
$$

If the plate is clamped along its edges, the appropriate boundary conditions can be written in the nondimensional form

$$
\begin{aligned}
U = V = W = W_{,\zeta} = 0 \quad &\text{at} \quad \zeta = \pm 1 \\
U = V = W = W_{,\eta} = 0 \quad &\text{at} \quad \eta = \pm 1.
\end{aligned} \tag{6}
$$

Equations (3) are to be solved in conjunction with boundary conditions (6).

SOLUTION

The parameters in load, deflection and two in-plane displacements are developed into power series with respect to the nondimensional central deflection, $W(0, 0)$, of the plate, denoted by W_0.

$$
\begin{aligned}
Q &= \sum_{n=1}^{\infty} q_n W_0^n, \qquad & W &= \sum_{n=1}^{\infty} w_n(\zeta, \eta) W_0^n \\
U &= \sum_{n=1}^{\infty} u_n(\zeta, \eta) W_0^n, \qquad & V &= \sum_{n=1}^{\infty} v_n(\zeta, \eta) W_0^n.
\end{aligned} \tag{7}
$$

By definition, it requires that

$$
w_1(0, 0) = 1, \qquad w_i(0, 0) = 0, \qquad i = 2, 3, \ldots \tag{8}
$$

Substituting equations (7) into (3) and (6) and equating like powers of W_0, a series of differential equations and boundary conditions are obtained. In the first approximation, the terms in the first power of W_0 are equated. The corresponding set of differential equations are given by

$$
\begin{aligned}
\lambda L_1 u_1 + \lambda L_2 v_1 - L_3 w_1 &= 0 \\
\lambda L_2 u_1 + \lambda L_4 v_1 - L_5 w_1 &= 0 \\
\lambda L_3 u_1 + \lambda L_5 v_1 - L_6 w_1 &= -\lambda^4 q_1
\end{aligned} \tag{9}
$$

which governs the small deflection of unsymmetrically laminated anisotropic plates. The second approximation leads to

$$
\begin{aligned}
\lambda L_1 u_2 + \lambda L_2 v_2 - L_3 w_2 &= -w_{1,\zeta} L_1 w_1 - \lambda w_{1,\eta} L_2 w_1 \\
\lambda L_2 u_2 + \lambda L_4 v_2 - L_5 w_2 &= -w_{1,\zeta} L_2 w_1 - \lambda w_{1,\eta} L_4 w_1 \\
\lambda L_3 u_2 + \lambda L_5 v_2 - L_6 w_2 &= -\lambda^4 q_2 - \lambda u_{1,\zeta} L_7 w_1 - \lambda^2 v_{1,\eta} L_8 w_1 \\
&\quad - \lambda(\lambda u_{1,\eta} + v_{1,\zeta}) L_9 w_1 - w_{1,\zeta} L_3 w_1 - \lambda w_{1,\eta} L_5 w_1 \\
&\quad - 2\lambda(b_3 - b_4)(w_{1,\zeta\eta}^2 - w_{1,\zeta\zeta} w_{1,\eta\eta}).
\end{aligned} \tag{10}
$$

The third approximation yields

$$
\begin{aligned}
\lambda L_1 u_3 + \lambda L_2 v_3 - L_3 w_3 &= -w_{1,\zeta} L_1 w_2 - w_{2,\zeta} L_1 w_1 - \lambda(w_{1,\eta} L_2 w_2 + w_{2,\eta} L_2 w_1) \\
\lambda L_2 u_3 + \lambda L_4 v_3 - L_5 w_3 &= -w_{1,\zeta} L_2 w_2 - w_{2,\zeta} L_2 w_1 - \lambda(w_{1,\eta} L_4 w_2 + w_{2,\eta} L_4 w_1) \\
\lambda L_3 u_3 + \lambda L_5 v_3 - L_6 w_3 &= -\lambda^4 q_3 - \lambda u_{1,\zeta} L_7 w_2 - (\lambda u_{2,\zeta} + w_{1,\zeta}{}^2/2) L_7 w_1 \\
&\quad - \lambda^2 v_{1,\eta} L_8 w_2 - \lambda^2 (v_{2,\eta} + w_{1,\eta}{}^2/2) L_8 w_1 \\
&\quad - \lambda(\lambda u_{1,\eta} + v_{1,\zeta}) L_9 w_2 - \lambda(\lambda u_{2,\eta} + v_{2,\zeta} + w_{1,\zeta} w_{1,\eta}) L_9 w_1 \\
&\quad - w_{1,\zeta} L_3 w_2 - w_{2,\zeta} L_3 w_1 - \lambda(w_{1,\eta} L_5 w_2 + w_{2,\eta} L_5 w_1) \\
&\quad - 2\lambda(b_3 - b_4)(2w_{1,\zeta\eta} w_{2,\zeta\eta} - w_{1,\zeta\zeta} w_{2,\eta\eta} - w_{2,\zeta\zeta} w_{1,\eta\eta}).
\end{aligned} \tag{11}
$$

The other sets of differential equations in high-order approximations can be similarly obtained. The boundary conditions in any approximations are obtained from equations (6) and (7).

$$
\begin{aligned}
&\text{Along } \zeta = \pm 1: \quad u_n = v_n = w_n = w_{n,\zeta} = 0 \\
&\text{Along } \eta = \pm 1: \quad u_n = v_n = w_n = w_{n,\eta} = 0
\end{aligned}
\qquad n = 1, 2, \ldots \tag{12}
$$

In equations (7), q_1, w_1, u_1 and v_1 are then determined from equations (9) and (12) for $n = 1$, q_2, w_2, u_2 and v_2 from equations (10) and (12) for $n = 2$, and so forth. Once the solution of equations (9) are found, the right members of equations (10) become the known functions of ζ and η. Hence equations (10) result in a system of linear partial differential equations. By successive substitutions, any set of equations in other approximations are to be systems of linear equations.

The solution to any set of the foregoing differential equations is assumed to be in the form of polynomial

$$
\begin{aligned}
w_n &= (1-\zeta^2)^2(1-\eta^2)^2 F(\zeta, \eta) \\
u_n &= (1-\zeta^2)(1-\eta^2) G(\zeta, \eta) \qquad n = 1, 2, 3, \ldots \\
v_n &= (1-\zeta^2)(1-\eta^2) H(\zeta, \eta).
\end{aligned} \tag{13}
$$

In equations (13), F, G, and H are the complete polynomials defined by

$$
\begin{aligned}
F &= R_{n00} + R_{n10}\zeta + R_{n01}\eta + \cdots + R_{n60}\zeta^6 + R_{n51}\zeta^5\eta + R_{n42}\zeta^4\eta^2 + R_{n33}\zeta^3\eta^3 \\
&\quad + R_{n24}\zeta^2\eta^4 + R_{n15}\zeta\eta^5 + R_{n06}\eta^6 \\
G &= S_{n00} + S_{n10}\zeta + S_{n01}\eta + \cdots + S_{n50}\zeta^5 + S_{n41}\zeta^4\eta + S_{n32}\zeta^3\eta^2 + S_{n23}\zeta^2\eta^3 \\
&\quad + S_{n14}\zeta\eta^4 + S_{n05}\eta^5 \\
H &= T_{n00} + T_{n10}\zeta + T_{n01}\eta + \cdots + T_{n50}\zeta^5 + T_{n41}\zeta^4\eta + T_{n32}\zeta^3\eta^2 + R_{n23}\zeta^2\eta^3 \\
&\quad + T_{n14}\zeta\eta^4 + T_{n05}\eta^5
\end{aligned} \tag{14}
$$

where R's, S's, and T's are constant coefficients. In view of equations (8),

$$
R_{100} = 1, \qquad R_{n00} = 0, \qquad n = 2, 3, \ldots \tag{15}
$$

It is observed that equations (13) satisfy all the boundary conditions (12) for any value of n. Upon substitution of equations (13) for $n = 1$ in equations (9), twenty-three algebraic equations are generated from each of the first two of equations (9) and twenty-four equations from the last by equating coefficients of like powers of ζ and η to zero and the constant term on the left-hand side of the last equation to q_1. These linear algebraic equations except the one containing q_1 are then solved simultaneously for 69 coefficients in w_1, u_1 and v_1. By substitution, q_1 is determined. The procedure on the determination of the unknowns in the second and high order approximations is quite similar to that used in the first approximation.

Based on the previous work[13, 14], the series solution given by equations (7) may be taken to be

$$Q \approx \sum_{n=1}^{3} q_n W_0{}^n, \qquad W \approx \sum_{n=1}^{3} w_n(\zeta, \eta) W_0{}^n$$
$$U \approx \sum_{n=1}^{3} u_n(\zeta, \eta) W_0{}^n, \qquad V \approx \sum_{n=1}^{3} v_n(\zeta, \eta) W_0{}^n. \tag{16}$$

Once the unknowns, q_n, w_n, u_n and v_n, in equations (16) are determined, the nondimensional membrane stress resultants and stress couples can be, in view of equation (1), calculated from the following:

$$\begin{aligned} N_\zeta &= \delta_1 + a_6 \delta_2 + a_1 \delta_3 - b_1 W_{,\zeta\zeta}/\lambda^2 - b_3 W_{,\eta\eta} - 2b_2 W_{,\zeta\eta}/\lambda \\ N_\eta &= a_6 \delta_1 + a_5 \delta_2 + a_4 \delta_3 - b_3 W_{,\zeta\zeta}/\lambda^2 - b_6 W_{,\eta\eta} - 2b_5 W_{,\zeta\eta}/\lambda \\ N_{\zeta\eta} &= a_1 \delta_1 + a_4 \delta_2 + a_2 \delta_3 - b_2 W_{,\zeta\zeta}/\lambda^2 - b_5 W_{,\eta\eta} - 2b_4 W_{,\zeta\eta}/\lambda \end{aligned} \tag{17}$$

and

$$\begin{aligned} M_\zeta &= b_1 \delta_1 + b_3 \delta_2 + b_2 \delta_3 - c_1 W_{,\zeta\zeta}/\lambda^2 - c_6 W_{,\eta\eta} - 2c_2 W_{,\eta\eta}/\lambda \\ M_\eta &= b_3 \delta_1 + b_6 \delta_2 + b_5 \delta_3 - c_6 W_{,\zeta\zeta}/\lambda^2 - c_5 W_{,\eta\eta} - 2c_4 W_{,\eta\eta}/\lambda \\ M_{\zeta\eta} &= b_2 \delta_1 + b_5 \delta_2 + b_4 \delta_3 - c_2 W_{,\zeta\zeta}/\lambda^2 - c_4 W_{,\eta\eta} - (c_3 - c_6) W_{,\eta\eta}/\lambda \end{aligned} \tag{18}$$

where

$$\begin{aligned} (N_\zeta, N_\eta, N_{\zeta\eta}) &= (N_x, N_y, N_{xy}) b^2/h^2 A_{11} \\ (M_\zeta, M_\eta, M_{\zeta\eta}) &= (M_x, M_y, M_{xy}) b^2/h^3 A_{11} \\ \delta_1 &= (\lambda U_{,\zeta} + W_{,\zeta}{}^2/2)/\lambda^2, \qquad \delta_2 = V_{,\eta} + W_{,\eta}{}^2/2 \\ \delta_3 &= (\lambda U_{,\eta} + V_{,\zeta} + W_{,\zeta} W_{,\eta})/\lambda, \qquad c_6 = D_{12}/h^2 A_{11}. \end{aligned} \tag{19}$$

The solution presented above is simplified for some special cases.

In the case of unsymmetrical angle-ply plates, it can be shown that

$$A_{16} = A_{26} = D_{16} = D_{26} = 0 \tag{20}$$

and that all elements of the B matrix vanish except B_{16} and B_{26}. In view of equations (4), it follows that

$$\begin{aligned} a_1 &= a_4 = c_2 = c_4 = 0 \\ b_1 &= b_3 = b_4 = b_6 = 0. \end{aligned} \tag{21}$$

It is observed that the deflection and in-plane displacements in this case possess the following properties:

$$\begin{aligned} &w_n(-\zeta, -\eta) = w_n(\zeta, n), \qquad w_n(\zeta, -\eta) = w_n(-\zeta, \eta) \\ &u_n(-\zeta, -\eta) = -u_n(\zeta, \eta), \qquad u_n(\zeta, -\eta) = -u_n(-\zeta, \eta) \\ &v_n(-\zeta, -\eta) = -v_n(\zeta, \eta), \qquad v_n(\zeta, -\eta) = -v_n(-\zeta, \eta) \\ &u_n(0, 0) = v_n(0, 0) = 0 \qquad n = 1, 2, 3. \end{aligned} \tag{22}$$

To satisfy equations (22), we take in equations (14)

$$\begin{aligned} &R_{n10} = R_{n01} = R_{n30} = R_{n21} = R_{n12} = R_{n03} = 0 \\ &R_{n50} = R_{n41} = R_{n32} = R_{n23} = R_{n14} = R_{n05} = 0 \\ &Z_{n00} = Z_{n20} = Z_{n11} = Z_{n02} = Z_{n40} = Z_{n31} = 0 \\ &Z_{n22} = Z_{n13} = Z_{n04} = 0 \qquad n = 1, 2, 3 \end{aligned} \tag{23}$$

in which

$$Z_{nij} = S_{nij} \quad \text{or} \quad T_{nij}. \tag{24}$$

For unsymmetrical cross-ply plates, it is found that

$$\begin{aligned} &A_{16} = A_{26} = D_{16} = D_{26} = 0 \\ &A_{22} = A_{11}, \qquad B_{22} = -B_{11}, \qquad D_{22} = D_{11} \end{aligned} \tag{25}$$

and that all other elements of the B matrix vanish. Thus, from equations (4)

$$\begin{aligned} &a_1 = a_4 = b_2 = b_3 = b_4 = b_5 = 0 \\ &c_2 = c_4 = 0, \qquad a_5 = 1, \qquad b_6 = -b_1, \qquad c_5 = c_1. \end{aligned} \tag{26}$$

To satisfy the two-fold symmetry with respect to the xz- and yz-planes, we take in addition to equations (23)

$$\begin{aligned} &R_{n11} = R_{n31} = R_{n13} = R_{n51} = R_{n33} = R_{n15} = 0 \\ &S_{n01} = S_{n21} = S_{n03} = S_{n41} = S_{n23} = R_{n05} = 0 \\ &T_{n10} = T_{n30} = T_{n12} = T_{n50} = T_{n32} = T_{n14} = 0. \end{aligned} \tag{27}$$

In the case of symmetrically laminated anisotropic plates, the material coupling phenomenon does not occur between transverse bending and in-plane stretching. Consequently,

$$B_{ij} = 0, \qquad i, j = 1, 2, 6$$

or

$$b_i = 0, \qquad i = 1, 2, \ldots, 6. \tag{28}$$

In view of equations (5) and on the consideration of the geometric symmetry with respect to the middle plane, obtained is the following:

$$L_3 = L_5 = u_1 = v_1 = w_2 = u_3 = v_3 = 0. \tag{29}$$

For material homogeneity, equations (2) become

$$A_{ij} = hC_{ij} \quad \text{and} \quad D_{ij} = h^3 C_{ij}/12, \qquad i, j = 1, 2, 6 \tag{30}$$

in addition to equations (28) and (29). The general solution presented in this work, therefore, can reduce to those for anisotropic, orthotropic, and isotropic plates[13–16].

NUMERICAL RESULTS AND DISCUSSION

In numerical examples the solution (16) is applied to unsymmetrical angle-ply and cross-ply plates. In these two cases, the nondimensional parameters in uniform load, membrane stress resultants and stress couples defined in equations (4) and (19) can be simplified by replacing A_{11} by $E_T h$, in which E_T is the tensile modulus of an orthotropic material perpendicular to the filament direction. Calculations are performed for glass-epoxy, boron-epoxy, and graphite-epoxy composites. The elastic constants typical of these materials are listed in Table 1, where E_L is the tensile modulus in the filament direction, G_{LT} is the shear modulus and ν_{LT} is the Poisson's ratio. In the case of angle-ply plates, the orthotropic axes of layers are oriented alternately at angles of 45° and −45° to the plate axes. The classical stiffness transformation is used to obtain $C_{ij}^{(k)}$. The total number of layers in a plate, denoted by n, is taken to be 2, 4, 6 and ∞ and the aspect ratio λ, to be 1·0, 1·5 and 2·0. To avoid confusion, it is best to regard b as fixed in magnitude and to think of large λ's as representing large values of a, whereas small λ's correspond to small values of a. In the presentation of

Table 1. Numerical values of elastic constants

	E_L/E_T	G_{LT}/E_T	ν_{LT}
Glass-epoxy	3·0	0·60	0·25
Boron-epoxy	10·0	1/3	0·22
Graphite-epoxy	40·0	0·50	0·25

numerical results, unsymmetrical angle-ply and cross-ply plates are respectively denoted by *AP* and *CP* and glass-epoxy, boron-epoxy, and graphite-epoxy composites, by *GL*, *BO* and *GR*, respectively.

The validity of the three-term approximation given by equation (16) is examined by comparison with the two-term approximation. In the case of a $\pm 45°$ angle-ply two-layer graphite-epoxy plate, the load-deflection relations given by the one-term, two-term and three-term solutions are shown in Fig. 1 for $\lambda = 1{\cdot}0$, $1{\cdot}5$ and $2{\cdot}0$. It is seen from the figure

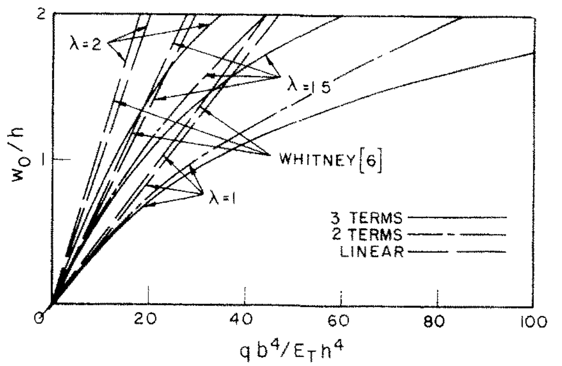

Fig. 1. Effect of aspect ratio on load-deflection relation for $\pm 45°$ angle-ply two-layer graphite-epoxy plate.

that the series given by equation (7) converges rapidly and that the values of the central deflection w_0 given by the two-term and three-term solutions are subjected to a maximum difference of 5 per cent for the central deflection equal to the plate thickness. For the plate with $n = 4$, the numerical result (not presented here) shows that the three-term approximation can be applied to $w_0 = 1{\cdot}1\ h$ within the same accuracy. Generally the three-term solution is valid in a range of deflections larger for glass-epoxy and boron-epoxy plates than for the equivalent graphite-epoxy plates. In this analysis all the numerical results are based on the three-term approximation and no attempt is made to use the higher order approximations.

The relation between transverse load q and central deflection w_0 of a two-layer graphite-epoxy plate is shown in Fig. 1 for various aspect ratios. It is observed that for a given load the central deflection of the plate increases with the aspect ratio. This is also true for cross-ply plates. The linear load-deflection relations are also presented and agree fairly with the graphical results given by Whitney[6]. Figure 2 shows the variation of the central deflection of a square two-layer plate with the transverse load for different material properties. When the deflection is held constant, a high pressure is required for a high-modulus material. For a fixed load the central deflection of an angle-ply plate is slightly greater than an equivalent cross-ply plate. In Fig. 3 the load-deflection curves are presented for various values of the

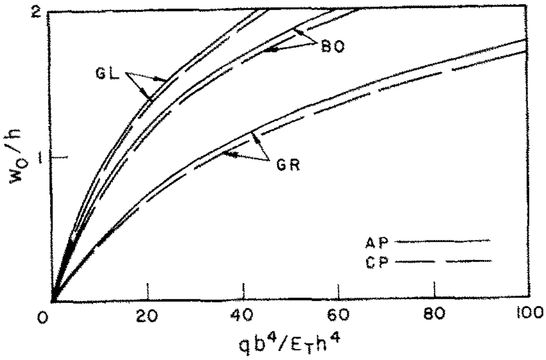

Fig. 2. Effect of material properties on load-deflection relations for square two-layer cross-ply and $\pm 45°$ angle-ply plates.

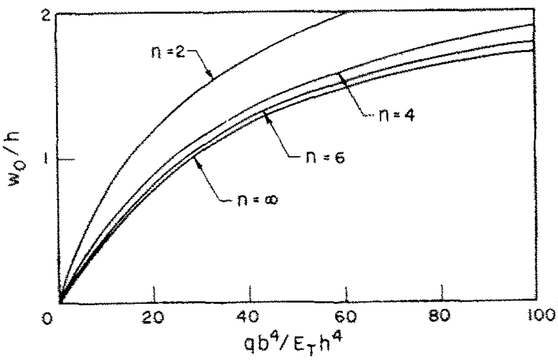

Fig. 3. Effect of number of layers on load-deflection relation for square $\pm 45°$ angle-ply boron-epoxy plate.

total number of layers in a square angle-ply boron-epoxy plate. For fixed values of plate thickness and load, the central deflection decreases as the total number of layers increases. The curve for $n = 6$ is close to that given by the uncoupled solution ($n = \infty$) for which $B_{ij} = 0$.

Figures 4 and 5 respectively show the stress couple M_ξ at the center of a plate and at the

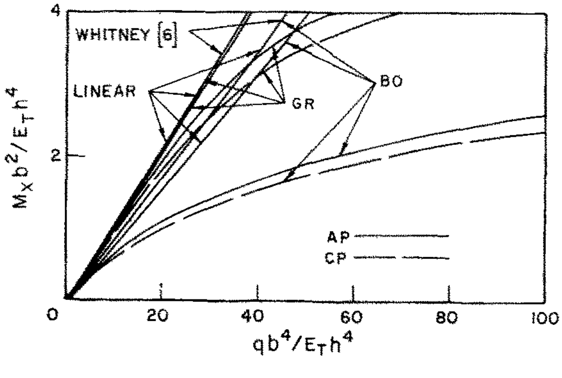

Fig. 4. Stress couple M_x at the center of square four-layer cross-ply and $\pm 45°$ angle-ply plates of different materials.

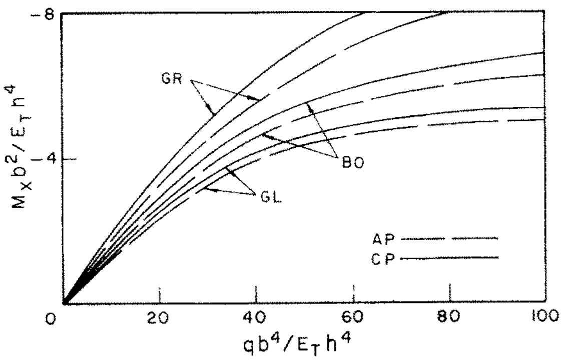

Fig. 5. Stress couple M_x at edge midpoint (1, 0) of square two-layer $\pm 45°$ angle-ply plate of different materials.

midpoint of a transverse edge. The results are presented for both cross-ply and angle-ply laminates of various materials. For a given pressure a large stress couple in magnitude occurs for a high-modulus material. The results based on the linear theory are also shown in Fig. 4. The present value is in good agreement with Whitney's result[6] for the graphite-epoxy plate but not for the boron-epoxy laminate. This is because the shear modulus and Poisson's ratio for the boron-epoxy plate are different from those used by Whitney. Finally, the membrane stress resultant N_ξ at different points are presented in Fig. 6 for a square

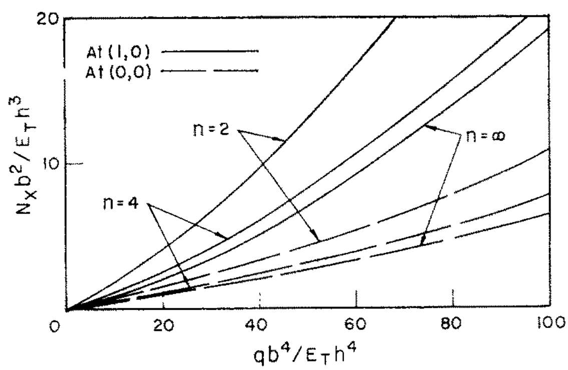

Fig. 6. Membrane stress resultant N_x at the center (0, 0) and edge midpoint (1, 0) of square cross-ply boron-epoxy plate for various values of number of layers.

cross-ply boron-epoxy plate. It is found that for a given load the membrane stress resultant at a point decreases with increasing the total number of layers and the largest value occurs at the midpoint of a plate edge as the stress couple M_ξ.

CONCLUSIONS

A series solution is formulated for the large deflection of a clamped unsymmetrically laminated anisotropic plate under uniform transverse load. This solution can reduce to some existing solutions for large deflections of isotropic, orthotropic and anisotropic plates[13–16]. The three-term approximation used in calculations for unsymmetrical cross-ply and angle-ply plates is illustrated to be applicable to the central deflection equal to the thickness of plate. For deflections larger than the plate thickness, higher order approximations are recommended. The present results in small deflections of unsymmetrical laminates agree closely with available data[6]. Some concluding remarks regarding the large deflection behavior of unsymmetrical cross-ply and angle-ply plates may be drawn from the present study.

When the central deflection is held constant, a large load is required for a high-modulus material. For a fixed pressure, the central deflection increases with the aspect ratio but decreases with the number of layers in the plate. As in the case of small deflections, coupling between bending and stretching reduces the effective stiffness of the plate.

The uncoupled solution is a good approximation for large deflections, stress couples and stress resultants of a coupled plate consisting of a large number of layers ($n \geq 6$) as in the case of small deflections. However, the coupling effect for large deflections of two-layer laminates is not so significant as that for small deflections.

The central deflection, stress couple M_x and membrane stress resultant N_x of an angle-ply plate are greater than those of the corresponding cross-ply plate. Both the stress couple and stress resultant are of the largest magnitudes at the midpoint of a transverse edge as in the case of homogeneous plates

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Абстракт — Дается приближенные решения для больших прогибов защемленных, равномерно нагруженных, несиметрически слоистных, анизотропных, прямоугольных пластинок. Выражая нагрузку и перемещения в форме рядов, сводятся нелинейные дифференциальные уравнения типа Кармана и неподвижные граничные условия к рядам линейных дифференциальных уравнений в частных производных и граничным условиям. Полученное решение путем последовательных приближений можно приводить к некоторым существующим решениям для больших прогибов однородных пластинок. Даются графически численные результаты на основе трех первых членов усеченных рядов для пластинок с несимметрическим поперечным слоем угловым слоем; эти пластинки обладают разными значениями материала для усиления волокнами, числом слоев и отношением положения. Сравниваются результаты в области малых протибов для спаренных слоев с доступными данными.